TEMPERATURE EVOLUTION LAW OF IMPERFECT RELATIVISTIC FLUIDS

R. Silva * J. A. S. Lima† and M. O. Calvão‡

Departamento de Física Teórica e Experimental

Universidade Federal do Rio Grande do Norte, C. P. 1641

59072-970, Natal-RN, Brazil

[2]*Instituto de Física,*

Universidade Federal do Rio de Janeiro,

21945-970, Rio de Janeiro, RJ, Brazil

Abstract

The first-order general relativistic theory of a generic dissipative (heat-conducting, viscous, particle-creating) fluid is rediscussed from a unified covariant frame-independent point of view. By generalizing some previous works in the literature, we derive a formula for the temperature variation rate, which is valid both in Eckart's (particle) and in the Landau-Lifshitz (energy) frames. Particular attention is paid to the case of gravitational particle creation and its possible cross-effect with the bulk viscosity mechanism.

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*Electronic address: raimundo@dfte.ufrn.br

†Electronic address: limajas@dfte.ufrn.br

‡Electronic address: orca@if.ufrj.br

I. INTRODUCTION

The formulation of the cosmological problem, for any particular model, is based on a set of phenomenological quantities such as energy density, pressure and temperature, which in principle, need to be defined with respect to local observers. In this regard, as pointed out by Géheniau and collaborators [1], the interdisciplinary field of cosmology is the natural inheritor of the concepts and methods ordinarily employed in hydrodynamics or, more generally, in non-equilibrium thermodynamics [2,3].

Besides being essential to complete the aforementioned phenomenological description, the thermodynamic analysis of cosmological models may also have interesting observational consequences. In principle, the dimensionless entropy content of the (observable) universe, a huge number of the order of 10^{87}, may have been generated by dissipative mechanisms like heat conduction, viscosity, "chemical reactions", diffusion, photon creation, etc., taking place in an inhomogeneous and anisotropic stage of the early universe [4,5]. In this case, after some few minutes, the gradients of the main physical quantities will be constrained from primordial nucleosynthesis studies. Later on, at the time of recombination, these irreversible mechanisms may also contribute to the temperature anisotropy of the cosmic background radiation. Indeed, although severely restricted from COBE measurements, the presence of the classical dissipative mechanism are naturally expected in a non-homogeneous quasi FRW universe, and, in principle, their specific signature must be somewhat different of the standard Sunyaev-Zeldovich and Sachs-Wolf effects [6].

In order to unify in a single and coherent scheme all irreversible phenomena occurring in a simple fluid or in mixtures, standard non-equilibrium thermodynamics works with two basic ideas [2,3]. The first one is the local equilibrium hypothesis, whose mathematical expression is given by the equilibrium Gibbs law in its local form. It implies that, out of equilibrium, the basic state functions such as the entropy, depend locally on the same set of thermodynamic variables as in equilibrium. In particular, the usual thermodynamic temperature and pressure concepts are maintained in the non-equilibrium regime. The

second idea is that, in the presence of dissipative processes, there is a local entropy source strength τ (entropy variation per unit volume and unit time), which, by the second law of thermodynamics, is always non-negative. Mathematically, it takes the form of a balance equation with τ as a source term.

By combining such assumptions with the fluid equations of motion, one finds an expression for the entropy source strength, as well as for the constitutive (phenomenological) relations themselves. The usual equilibrium theory is readily recovered by taking the limit of vanishing entropy production rate.

This approach has systematically been applied at the classical level [2,3] as well as in the special-relativistic domain [5]- [9]. Its extension to the general-relativistic framework is straightforward, provided the gravitational field varies slowly over the mean free path or the mean free time of the fluid particles [4]. Formally, since, under these conditions, the equivalence principle should hold, the generally covariant equations may be established by the usual minimal coupling recipe, that is, the replacement of usual derivatives by covariant derivatives and the replacement of the Minkowski metric $\eta_{\alpha\beta}$ by its Riemannian counterpart $g_{\alpha\beta}$. However, as is widely known, in the relativistic case, there is an ambiguity related to the possible choices of the macroscopic hydrodynamic four-velocity. In Eckart's formulation [7], the four-velocity is directly related to the particle flux, while, in Landau-Lifshitz's approach [8], it is directly related to the energy flux. In principle, a general treatment should be able to deal with any of these "gauge" choices.

In this paper, we consider a unified covariant description of a heat-conducting viscous simple fluid with particle creation. The theories of Eckart and Landau-Lifshitz will be seen to be special cases of this general formulation. We also derive a new temperature evolution law which holds for any particular choice of the macroscopic hydrodynamic four-velocity.

II. GENERAL RELATIVISTIC THEORY OF FLUIDS

The thermodynamic state of a relativistic simple fluid is characterized by an energy-momentum tensor $T^{\alpha\beta}$, a particle current N^α and an entropy current S^α. The fundamental equations of motion are expressed by the conservation law (semi-colon denotes covariant derivative) of energy-momentum and the equation of balance for the particle number [10]

$$T^{\alpha\beta}{}_{;\beta} = 0 \quad , \tag{1}$$

$$N^\alpha{}_{;\alpha} = \Psi \quad , \tag{2}$$

where Ψ is a particle source ($\Psi > 0$) or sink ($\Psi < 0$) term. The second law of thermodynamics requires that the entropy source strength be non-negative

$$S^\alpha{}_{;\alpha} = \tau \geq 0 \quad , \tag{3}$$

where $\tau = 0$ describes a non-dissipative state, and $\tau > 0$ denotes a dissipative state. A perfect fluid always evolves through non-dissipative (equilibrium) states, whereas an imperfect fluid typically evolves through dissipative (non-equilibrium) states (see, however, [11]).

A. Adiabatic Limit

By choosing an arbitrary hydrodynamic frame of reference, whose four-velocity obeys $u^\alpha u_\alpha = 1$, the primary variables $T^{\alpha\beta}, N^\alpha$ and S^α take the following forms [5,9]

$$T^{\alpha\beta} = \rho u^\alpha u^\beta - p h^{\alpha\beta}, \tag{4}$$

$$N^\alpha = n u^\alpha, \tag{5}$$

$$S^\alpha = n\sigma u^\alpha, \tag{6}$$

where the tensor $h^{\alpha\beta} := u^\alpha u^\beta - g^{\alpha\beta}$ is the usual projector onto the local rest space of u^α. The variables ρ, p, n and σ stand respectively for the energy density, thermostatic pressure,

particle number density and specific entropy (per particle), and are related by the so-called Gibbs law [5,9]

$$nTd\sigma = d\rho - \frac{\rho + p}{n} dn \quad . \tag{7}$$

As a particular result of the general theory, (4)-(6) imply that the entropy source strength defined by (3) vanishes identically,

$$S^{\alpha}{}_{;\alpha} = \tau = 0 \quad , \tag{8}$$

as should be expected for thermal equilibrium states.

B. Non-Equilibrium States

In principle, the inclusion of dissipative processes such as heat conduction and viscosity, requires additional terms in the primary variables describing a perfect fluid. However, unlike in the adiabatic limit case, the presence of a heat transfer poses a problem regarding the definition of the hydrodynamic four-velocity u^{α}. It is necessary to specify whether u^{α} is the four-velocity of the energy transport or particle transport. In Eckart's formulation, u^{α} is identified with the four-velocity of particle transport (particle frame) [7]. In the approach of Landau-Lifshitz, u^{α} is defined as the four-velocity of energy transport (energy frame) [8]. Formally, the particle frame is the unique unit time-like vector parallel to N^{α}, whereas the energy frame is the unique unit time-like eigenvector of $T^{\alpha\beta}$. Both theories assume that, for weak space-time gradients, the basic quantities contain no terms higher than first order in deviations from equilibrium.

In the presence of irreversible processes, we must add small terms $\Delta T^{\alpha\beta}$ and ΔN^{α} in (4) and (5), which are restricted by the second law of thermodynamics (3)

$$T^{\alpha\beta} = \rho u^{\alpha} u^{\beta} - p h^{\alpha\beta} + \Delta T^{\alpha\beta} \quad , \tag{9}$$

$$N^{\alpha} = n u^{\alpha} + \Delta N^{\alpha} \quad , \tag{10}$$

Usually, at this point, one specifies whether the Eckart or Landau-Lifshitz approach will be adopted. However, this is not necessary in the covariant frame-independent formulation presented here since these theories will be recovered as particular cases.

As before the fluid motion equations are contained in (1) and (2), which express the energy conservation law and the balance equation for the particle number, respectively. Therefore, differentiating the expression (9) and projecting it on the direction of the four-velocity u_α, one finds

$$u_\alpha T^{\alpha\beta}{}_{;\beta} = \dot{\rho} + (\rho + p)\theta + u_\alpha \Delta T^{\alpha\beta}{}_{;\beta} = 0 \quad , \tag{11}$$

and from (2)

$$N^\alpha{}_{;\alpha} = \dot{n} + n\theta + \Delta N^\alpha{}_{;\alpha} = \Psi \quad , \tag{12}$$

where an overdot means the derivative along the world lines of the fluid volume element, e.g., $\dot{\rho} = u^\beta \rho_{;\beta}$, and $\theta = u^\beta{}_{;\beta}$ is the expansion rate of the fluid.

Now, taking the covariant derivative of (7) along the world lines of the fluid volume element, and making use of equations (11) and (12), we obtain

$$T(n\sigma u^\beta)_{;\beta} = -u^\alpha \Delta T^{\alpha\beta}{}_{;\beta} - \mu\Psi + \mu\Delta N^\beta{}_{;\beta} \quad , \tag{13}$$

where μ is chemical potencial defined by Euler's relation

$$\mu = \frac{\rho + p}{n} - \sigma T \quad . \tag{14}$$

Defining the entropy flux

$$S^\beta = n\sigma u^\beta - \frac{\mu}{T}\Delta N^\beta + \frac{u_\alpha}{T}\Delta T^{\alpha\beta} \quad , \tag{15}$$

we see from (13) that the entropy source strength assumes the following form

$$S^\beta{}_{;\beta} = \left(\frac{u_{\alpha;\beta}}{T} - \frac{T_{;\beta}u_\alpha}{T^2}\right)\Delta T^{\alpha\beta} - \left(\frac{\mu_{;\beta}}{T} - \frac{\mu T_{;\beta}}{T^2}\right)\Delta N^\beta - \frac{\mu\Psi}{T} \quad , \tag{16}$$

which is a function dependent only on the dissipative fluxes.

At the level of the primary fluxes, the effect of the dissipative processes is to add up new fields for the energy-momentum tensor and the particle flux vector [7,5,10,13], which must be constrained by the second law. We have

$$\Delta T^{\alpha\beta} = -(\Pi + p_c)h^{\alpha\beta} + q^\alpha u^\beta + q^\beta u^\alpha + \Pi^{\alpha\beta} \quad , \tag{17}$$

and

$$\Delta N^\beta = \nu^\beta \quad , \tag{18}$$

where the five additional fields $\Pi, p_c, q^\alpha, \nu^\beta$ and $\Pi^{\alpha\beta}$ are, respectively, the bulk viscous pressure, the creation pressure (due to the gravitational matter creation [10]), the heat flow, the particle drift, and the shear viscosity stress. These fields describe the deviations from equilibrium within the fluid and satisfy the following constraints

$$u^\alpha q_\alpha = u^\alpha \nu_\alpha = u^\alpha \Pi_{\alpha\beta} = g_{\alpha\beta}\Pi^{\alpha\beta} = \Pi^{[\alpha\beta]} = 0 \quad , \tag{19}$$

where the square brackets denote anti-symmetrization and round brackets, below, symmetrization.

As one may check, inserting equations (17) and (18) into (16), and using (19), it is readily seen that

$$S^\alpha{}_{;\alpha} = -\frac{\Pi\theta}{T} - \frac{p_c\theta}{T} - \frac{\mu\Psi}{T} - \left(\frac{T_{;\beta}}{T^2} - \frac{\dot{u}_\beta}{T}\right) q^\beta - \left(\frac{\mu_{;\beta}}{T} - \frac{\mu T_{;\beta}}{T^2}\right) \nu^\beta + \frac{u_{\alpha;\beta}\Pi^{\alpha\beta}}{T} \tag{20}$$

On the other hand, the covariant derivative of u^α may be decomposed as [14]

$$u_{\alpha;\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta} + \dot{u}_\alpha u_\beta + \omega_{\alpha\beta} \quad , \tag{21}$$

where

$$\sigma_{\alpha\beta} = \frac{1}{2} h^\mu_\alpha h^\nu_\beta (u_{(\mu;\nu)} - \frac{2}{3}\theta h_{\mu\nu}) \quad , \tag{22}$$

and

$$\omega_{\alpha\beta} = \frac{1}{2} h^\mu_\alpha h^\nu_\beta u_{[\nu,\mu]} \quad , \tag{23}$$

are respectively the symmetric traceless shear tensor and the vorticity tensor. Inserting (21) into equation (20), we may rewrite the entropy source as

$$S^{\alpha}{}_{;\alpha} = \tau_s - \frac{q_\alpha h^{\alpha\beta}(T_{;\beta} - T\dot{u}_\beta)}{T^2} - \left(\frac{\mu_{;\beta}}{T} - \frac{\mu T_{;\beta}}{T^2}\right)\nu^\beta + \frac{\Pi^{\alpha\beta}\sigma_{\alpha\beta}}{T} \quad , \tag{24}$$

where τ_s describes the entropy source strength due to the scalar irreversible processes

$$\tau_s = -\frac{\Pi\theta}{T} - \frac{p_c\theta}{T} - \frac{\mu\Psi}{T} \quad . \tag{25}$$

Hence, to the first-order of approximation, (24) will be consistent with the second law of thermodynamics if the phenomenological relations among the vector and tensor dissipative fluxes and thermodynamic forces are taken to be

$$q^\alpha = \chi\phi^\alpha \quad ; \quad \phi^\alpha = h^{\alpha\beta}(T_{;\beta} - T\dot{u}_\beta) \quad , \tag{26}$$

$$\Pi^{\alpha\beta} = \eta\sigma^{\alpha\beta} \quad , \tag{27}$$

$$\nu^\alpha = \zeta\lambda^\alpha; \quad \lambda^\alpha = h^{\alpha\beta}\left(\frac{\mu}{T}\right)_{;\beta} \quad , \tag{28}$$

where χ, η, and ζ stand, respectively, for thermal conductivity, shear viscosity and "difusion" coefficients. Using (26)-(28), the entropy source strength (24) becomes

$$S^{\alpha}{}_{;\alpha} = \tau_s - \frac{\chi\phi_\alpha\phi^\alpha}{T^2} - \frac{\zeta\lambda_\alpha\lambda^\alpha}{T^2} + \frac{\eta\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{T} \quad , \tag{29}$$

and since the heat flow and the particle drift are space-like vectors ($\phi^\alpha\phi_\alpha < 0, \lambda^\alpha\lambda_\alpha < 0$), the second law of thermodynamics will be satisfied if, for any configuration of the fluid, χ, η and ζ are positive. In order to treat the scalar irreversible processes, we observe that they are fluxes with same tensor rank (scalars), and in the first-order of approximation this give rise to cross effects [2,15]. In order to describe this cross effect, we propose the following relations among fluxes and thermodynamic forces,

$$\Pi = -\xi_{11}\left(\frac{\Psi}{\theta}\right) - \xi_{12}\theta \quad , \tag{30}$$

and

$$p_c = -\xi_{21}\theta - \xi_{22}\left(\frac{\Psi}{\theta}\right) \quad , \tag{31}$$

where Π, p_c are fluxes and $\left(\frac{\Psi}{\theta}\right)$, θ are thermodynamic forces, with $\xi_{ij} > 0$ $(i, j = 1, 2)$ phenomenological coefficients. When the particle source strength Ψ is zero, we see from (30)-(31) that $\Pi = -\xi_{12}\theta$ and $p_c = -\xi_{21}\theta$. Recalling that the phenomenological coefficients are supposed to obey Onsager's reciprocity relations, it follows that $\xi_{12} = \xi_{21}$. This means that in this particular case (no particle creation) the creation pressure and bulk viscosity reduce to the same process. In the general case, the entropy source (29) can be written as

$$S^{\alpha}{}_{;\alpha} = -\frac{\chi\phi_\alpha\phi^\alpha}{T^2} + \frac{\eta\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{T} - \frac{\zeta\lambda_\alpha\lambda^\alpha}{T^2} + \frac{2\xi_{12}\theta^2}{T} + \frac{(\xi_{11} + \xi_{22} - \mu)\Psi}{T} \tag{32}$$

We see that the second law of thermodynamics will be satisfied for any configurations of the fluid if χ,η,ζ,ξ_{12} are positive definite. We also see that the remaining coefficients ξ_{11} and ξ_{22} must satisfy the following inequalities: $\xi_{11} + \xi_{22} > \mu$ if $\Psi > 0$, and $\xi_{11} + \xi_{22} < \mu$ if $\Psi < 0$ (cf. [10]).

Concerning the results presented in the reference [15], we have the following remarks. First, we observe that the flux associated with the particle creation rate, namely, the creation pressure was not considered. Second, the authors have used Π as a force and not as, we think, a thermodynamic flux. Another question is related to the hypothesis implicitly adopted in their paper, that the creation pressure and the bulk viscous pressure are the same ($\Pi = p_c$). In particular, from the phenomenological laws obtained there, we see that for photons ($\mu = 0$), the creation pressure is given by $p_c = \frac{\xi_{12}}{\xi_{22}}\Psi$; however for the static fluid $\theta = 0$, and using the energy conservation, we find that $\dot{\rho} = 0$. These results seem to be physically inconsistent, because we have energy density ρ constant and simultaneously photon creation, which should be responsible for a variation of ρ. Note also that, in the adiabatic limit, the entropy source strength vanishes, as expected (see 8).

Summing up, the basic set of equations governing the first-order theory of dissipative simple fluids in a general formulation are: the energy conservation law (1) and equation of

balance (2) with $T^{\alpha\beta}$ and N^{α} given by (9) and (10); the Gibbs law (7), the constitutive equations (26)-(28) and the constitutive equation for a scalar process (30)-(31) with the fluxes constrained by (19) and linked with the entropy source strength by (32). Notice that such a system is underdetermined since, in principle, there are fifteen independent equations and seventeen unknowns; more specifically, as in the classical theory, the time behavior of the relativistic one-component fluid can only be determined, for specified initial boundary conditions, by adding two equations of state, say

$$p = p(n, T) \quad , \tag{33}$$

and

$$\rho = \rho(n, T) \quad . \tag{34}$$

We notice that, although sucessful in revealing the physics underlying a large class of phenomena, the first-order theories present some experimental and theoretical drawbacks. In its classical version, the linear constitutive equations (26)-(28) and (30)-(31) are not adequate at high frequencies or short wave lengths as manifested in experiments on ultrasound propagation in rarefied gases and on neutron scattering in liquids [16]. Besides they also allow the propagation of perturbations with arbitrarily high speeds, which, although perhaps merely unsatisfactory on classical grounds, is completely unacceptable from a relativistic point of view; furthermore, they do not have a well-posed Cauchy problem and their equilibrium states are not stable. Several authors have formulated relativistic second-order theories which circumvent these defficiencies [9,17–19]. In a forthcoming paper [20], we intend to extend our considerations to this class of theories.

III. TEMPERATURE EVOLUTION LAW

Let us now derive the general equation for the temperature law in a relativistic fluid taking into account all the irreversible processes. In what follows we consider that the

energy density and the equilibium pressure are functions of the thermodynamic variables n and T. Differentiating $\rho = \rho(n, T)$ along the world lines of the fluid volume element one finds

$$\dot{\rho} = \left(\frac{\partial \rho}{\partial T}\right)_n \dot{n} + \left(\frac{\partial \rho}{\partial n}\right)_T \dot{T} \quad , \tag{35}$$

and combining with the energy conservation law (11) and the balance equation (12), we obtain

$$(\frac{\partial \rho}{\partial T})_n \dot{T} = \left[n (\frac{\partial \rho}{\partial n})_T - \rho - p \right] \theta - (\frac{\partial \rho}{\partial n})_T (\Psi - \Delta N^\alpha{}_{;\alpha}) - u_\alpha \Delta T^{\alpha\beta}{}_{;\beta} \quad . \tag{36}$$

Since $d\sigma$ must be an exact differential, (7) leads to the thermodynamic relation

$$\left[\frac{\partial}{\partial T} \left(\frac{1}{nT} \left[\left(\frac{\partial \rho}{\partial n} \right)_T - \left(\frac{\rho + p}{n} \right) \right] \right) \right]_n = \left[\frac{\partial}{\partial n} \left(\frac{1}{nT} \left(\frac{\partial \rho}{\partial T} \right)_n \right) \right]_T \quad , \tag{37}$$

or equivalently,

$$T \left(\frac{\partial p}{\partial T} \right)_n = \rho + p - n \left(\frac{\partial \rho}{\partial n} \right)_T \quad , \tag{38}$$

and combining (38) with (36), we obtain the general equation governing the variation of temperature

$$\frac{\dot{T}}{T} = - \left(\frac{\partial p}{\partial \rho} \right)_n \theta - \frac{1}{T \left(\frac{\partial \rho}{\partial T} \right)_n} \left[\left(\frac{\partial \rho}{\partial n} \right)_T (\Psi - \Delta N^\alpha{}_{;\alpha}) + u_\alpha \Delta T^{\alpha\beta}{}_{;\beta} \right] \quad , \tag{39}$$

where all dissipatives fluxes are described by ΔN^α and $\Delta T^{\alpha\beta}$. In the adiabatic limit, the above equation reduces to the temperature variation rate of a perfect fluid

$$\frac{\dot{T}}{T} = - \left(\frac{\partial p}{\partial \rho} \right)_n \theta \quad . \tag{40}$$

Consider now in (39) the Landau-Lifshitz or energy frame. In this case, the comoving observers do not see the irreversible contribution of the energy flux, that is, $q^\alpha = 0$ in the energy-momentum tensor. As we have seen, there is an additional contribution ν^α in the particle flux ΔN^α. We obtain

$$\frac{\dot{T}}{T} = -(\frac{\partial p}{\partial \rho})_n \theta_{LL} + \frac{1}{T (\frac{\partial \rho}{\partial T})_n} \left[(\frac{\partial \rho}{\partial n})_T (\nu^\alpha_{;\alpha} - \Psi) - (\Pi + p_c) \theta_{LL} + \sigma_{\alpha\beta} \Pi^{\alpha\beta} \right] \tag{41}$$

where

$$\theta_{LL} = \frac{\Psi - \dot{n} - \nu^{\alpha}{}_{;\alpha}}{n} \tag{42}$$

is the expansion of the fluid in the Landau-Lifshitz frame.

Now let us consider the Eckart or particle frame formulation. In this case, the comoving observers do not see the irreversible contribution of particle drift, that is $\Delta N^{\alpha} = \nu^{\alpha} = 0$ in the particle flux vector. We have

$$\frac{\dot{T}}{T} = -(\frac{\partial p}{\partial \rho})_n \theta_E + \frac{1}{T(\frac{\partial \rho}{\partial T})_n}\left[-(p_c + \Pi)\theta_E + \dot{u}_{\alpha} q^{\alpha} + \sigma_{\alpha\beta}\Pi^{\alpha\beta} - q^{\beta}{}_{;\beta} - (\frac{\partial \rho}{\partial n})_T \Psi\right] \tag{43}$$

where

$$\theta_E = \frac{\Psi - \dot{n}}{n} \tag{44}$$

is the expansion of the fluid in the Eckart frame.

It should be noticed that in the adiabatic limit, when all dissipative fluxes are absent, equations (41) and (43) reproduce the same temperature variation rate of a perfect fluid. The same happens when the dissipative fluxes reduce to a creation pressure plus the bulk viscosity mechanism. For this homogeneous and isotropic dissipative simple fluid, the temperature law has previously been derived by Lima and Germano [21] (see also Zimdahl [22] to the case of a two-fluid mixture). We also emphasize that, for each formulation, the complete temperature evolution equation must play an important role in problems of astrophysics and cosmological interest envolving the classical non-equilibrium mechanisms. In the cosmological domain, for instance, a natural application of equations (41) and (43), is related to the contributions of the irreversible processes to the anisotropy temperature of the cosmic background radiation during the decoupling time.

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